

On the relativistic mass function and averaging in cosmology

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The general relativistic description of cosmological structure formation is an important challenge from both the theoretical and the numerical point of views. In this paper we present a brief prescription for a general relativistic treatment of structure formation and a resulting mass function on galaxy cluster scales in a highly generic scenario. To obtain this we use an exact scalar averaging scheme together with the relativistic generalization of Zel'dovich's approximation (RZA) that serves as a closure condition for the averaged equations.

Keywords: Relativistic cosmology; cosmological mass function.

1. Introduction

It is reasonable to assume that a proper analytic model should accompany if not precede any N -body simulation attempt, since it gives us a deeper understanding of the physics behind the process considered. That was the case with the classical Zel'dovich approximation (ZA) [1] and the Press–Schechter (PS) mass function formula,[2] predictions of which were confirmed afterwards, to a plausible degree, by Newtonian N -body simulations. In this line of thought a relativistic form of ZA as a subclass of the first-order Lagrangian perturbation theory [3] has been systematically translated to the relativistic stage,[4] generalizing the pioneering proposal by Kasai.[5] In an ongoing work we concentrate on the generalization of the mass function in this relativistic framework. We build on earlier work on the generalization of the Newtonian mass function [6, 7] that essentially introduces the complete initial data set, i.e. not only the overdensity but the three scalar invariants of the velocity gradient, to describe collapsing structures. This framework contains attempts to generalize the PS framework to a triaxial anisotropic collapse, since it is in addition inhomogeneous.

2. Averaging in cosmology

Given a flow-orthogonal, synchronous foliation of space-time (that restricts the matter model to irrotational dust), the averaged evolution of a general inhomogeneous and restmass-preserving spatial domain is subject to an effective form of Fried-

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mann's differential equations:[8, 9]

$$\begin{aligned} \left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 &= \frac{8\pi G \varrho_{\text{eff}}^{\mathcal{D}}}{3} + \frac{\Lambda}{3} - \frac{k_{\mathcal{D}}}{a_{\mathcal{D}}^2} ; \\ \left(\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right) &= -\frac{4\pi G(\varrho_{\text{eff}}^{\mathcal{D}} + 3p_{\text{eff}}^{\mathcal{D}})}{3} + \frac{\Lambda}{3} ; \\ \dot{\varrho}_{\text{eff}}^{\mathcal{D}} + 3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)(\varrho_{\text{eff}}^{\mathcal{D}} + p_{\text{eff}}^{\mathcal{D}}) &= 0 , \end{aligned} \quad (1)$$

where $a_{\mathcal{D}}$ is the domain dependent scale factor defined as the cubic root of the domain's volume, and where the sources are defined as $\varrho_{\text{eff}}^{\mathcal{D}} = \langle \varrho \rangle_{\mathcal{D}} + \varrho_{\Phi}^{\mathcal{D}}$ for the actual matter source $\langle \varrho \rangle_{\mathcal{D}}$ and the extra backreaction density $\varrho_{\Phi}^{\mathcal{D}}$. Notice that backreaction is a result of averaging also the geometrical side of Einstein's equations. Averaging leads to an effective pressure $p_{\text{eff}}^{\mathcal{D}} = p_{\Phi}^{\mathcal{D}}$ (Note that the matter model is still dust and the chosen foliation of space-time is unchanged). The new backreaction sources are defined in terms of the backreaction variables $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{W}_{\mathcal{D}} := \langle \mathcal{R} \rangle_{\mathcal{D}} - \frac{6k_{\mathcal{D}}}{a_{\mathcal{D}}^2}$, where this latter is the deviation of the averaged scalar curvature $\langle \mathcal{R} \rangle_{\mathcal{D}}$ from the homogeneous curvature term. For the backreaction sources we have:

$$\begin{aligned} \varrho_{\Phi}^{\mathcal{D}} &:= -\frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} - \frac{1}{16\pi G} \mathcal{W}_{\mathcal{D}} ; \\ p_{\Phi}^{\mathcal{D}} &:= -\frac{1}{16\pi G} \mathcal{Q}_{\mathcal{D}} + \frac{1}{48\pi G} \mathcal{W}_{\mathcal{D}} , \end{aligned} \quad (2)$$

allowing for an interpretation of the backreaction sources in terms of an effective scalar field.[10] The kinematical backreaction term $\mathcal{Q}_{\mathcal{D}}$ is built from two extrinsic curvature invariants that are related to the kinematical invariants rate of expansion Θ and rate of shear $\sigma^2 := \sigma^i_j \sigma^j_i$, with the shear tensor components σ_{ij} :

$$\mathcal{Q}_{\mathcal{D}} = 2\langle \Pi \rangle_{\mathcal{D}} - \frac{2}{3}\langle \text{I} \rangle_{\mathcal{D}}^2 ; \quad \text{I} := \Theta ; \quad \Pi := \frac{1}{3}\Theta^2 - \sigma^2 .$$

Equations (1) and Friedmann's are the same up to the dependence on the averaging domain; they are strictly the same if we postulate that homogeneous sources $\varrho_h(t)$ and $p_h(t)$ describe the average evolution, as is conjectured in the standard model and proved to hold in Newtonian cosmology.[11] In general relativity this no longer holds true due to the non-conservation of the averaged curvature.[12]

3. Relativistic Zel'dovich approximation and its average

The effective equations (1) can be closed by providing a dynamical equation of state that relates the effective sources. The relativistic Zel'dovich approximation (RZA) [4] provides such a closure. It prescribes a perturbation ansatz for Cartan co-frames,

$$\boldsymbol{\eta}^a = \boldsymbol{\eta}_H^a + a(t)\mathbf{P}^a ; \quad a = 1, 2, 3 , \quad (3)$$

where $\boldsymbol{\eta}_H^a = \eta_{H_i}^a \mathbf{d}X^i := a(t)\boldsymbol{\eta}_H^a(t_i)$, $\eta_{H_i}^a := a(t)\delta_i^a$ describes the background deformation in the exact basis $\mathbf{d}X^i$, $a(t)$ obeys the standard Friedmann equations,

and the inhomogeneous deformation one-form fields $\mathbf{P}^a(t, X^k) = P_i^a \mathbf{d}X^i$ may be developed into a perturbation series.[13] In coordinate components and at first order, RZA has the form:

$${}^{\text{RZA}}\eta_i^a(X^k, t) := a(t) \left(\delta_i^a + \xi(t) \dot{P}_i^a(X^k, t_i) \right), \quad (4)$$

with $\xi(t_i) = 0$; $\dot{\xi}(t_i) = 1$. The kinematical backreaction functional for this Lagrangian deformation field can be calculated:[14]

$${}^{\text{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\dot{\xi}^2 (\gamma_1 + \xi \gamma_2 + \xi^2 \gamma_3)}{(1 + \xi \langle \text{I}_i \rangle_{\mathcal{I}} + \xi^2 \langle \text{II}_i \rangle_{\mathcal{I}} + \xi^3 \langle \text{III}_i \rangle_{\mathcal{I}})^2}.$$

We here defined a formal (‘initial’) average, normalized by the initial volume,

$$\langle \mathcal{A} \rangle_{\mathcal{I}} := \frac{1}{V_{\mathcal{D}_i}} \int_{\mathcal{D}} \mathcal{A} \sqrt{G} d^3 X; \quad V_{\mathcal{D}_i} = \int_{\mathcal{D}_i} \sqrt{G} d^3 X, \quad (5)$$

where $G := \det(G_{ij})$ is taken on initial data for the metric coefficients, $G_{ij}(X^k) := g_{ij}(X^k, t_i)$; the subscript i marks the initial data and the abbreviations stand for

$$\begin{aligned} \gamma_1 &= 2 \langle \text{II}_i \rangle_{\mathcal{I}} - \frac{2}{3} \langle \text{I}_i \rangle_{\mathcal{I}}^2 = \mathcal{Q}_{\mathcal{D}_i}; \\ \gamma_2 &= 6 \langle \text{III}_i \rangle_{\mathcal{I}} - \frac{2}{3} \langle \text{I}_i \rangle_{\mathcal{I}} \langle \text{II}_i \rangle_{\mathcal{I}}; \\ \gamma_3 &= 2 \langle \text{I}_i \rangle_{\mathcal{I}} \langle \text{III}_i \rangle_{\mathcal{I}} - \frac{2}{3} \langle \text{II}_i \rangle_{\mathcal{I}}^2. \end{aligned} \quad (6)$$

Note that, despite the approximation made, this functional is exact for special plane-symmetric inhomogeneities and for spatially flat LTB solutions.[14] The corresponding functional in Newtonian cosmology [6] has the same form but contains the *general* plane- and spherically-symmetric solutions.

4. Relativistic mass function

Using RZA as a closure condition neglects pressure, velocity dispersion and vorticity, which are most relevant at small scales, as collapse processes accelerate. Generalizations of the matter model are in progress. In the following we will use the assumption that all of the dark matter particles at $z = 0$ are part of the dark matter halos (following the Press–Schechter treatment), and that all of the dark matter halos were formed from Gaussian distributions in each of the initial invariants ($\langle \text{I} \rangle_{\mathcal{D}_i}$, $\langle \text{II} \rangle_{\mathcal{D}_i}$ and $\langle \text{III} \rangle_{\mathcal{D}_i}$) with variances equal to those of the expectation values of the variance of these invariants, respectively (cf. App.C.2, [6]). We will also ignore the cloud-in-cloud problem since it mostly affects the lower mass scale of the mass function. Let us introduce the following notation: $n(z, M_i)$: the number of halos of mass M_i at redshift z per unit volume; $\hat{n}(M_i)$: the number of halos today; $F_i(z, M_i)$: the probability that objects of a given mass collapsed until the redshift z . $F_i(z, M_i)$ is calculated by a Monte Carlo procedure as explained in the corresponding Newtonian work.[6, 7] Since the probability of collapse is calculated

independently for each mass scale, and because we assume that all of the mass at redshift $z = 0$ is part of the collapsed objects, we need to normalise $F_i(0, M_i)$ so that (assuming $M_i = \rho_H(t_{\text{in}}) \frac{4}{3}\pi(R_{\text{in}})^3$, where the superscript ‘in’ stands for initial):

$$\int_{R_l}^{R_u} \alpha F_i(0, R) dR = 1, \quad (7)$$

where R_l and R_u correspond to lower (5 Mpc/h) and upper (80 Mpc/h) co-moving cut-offs, respectively, and α is the normalisation factor. This normalisation allows us to calculate the probability density of collapse for the given mass scale under the condition that all the mass in collapsed structures today adds up to the total mass of the domain containing these collapsed objects:

$$\bar{F}_i(z, M_i) = \alpha F_i(z, M_i). \quad (8)$$

The number of collapsed objects in an arbitrary volume V_H is then given by:

$$n(z, M_i) = \bar{F}_i(z, M_i) \frac{\rho_H V_H}{M_i}, \quad (9)$$

where ρ_H is an average density (in our case the density of an EdS background Universe). Assuming that $M_i = \rho_H(t_{\text{in}}) \frac{4}{3}\pi(R_{\text{in}})^3$, we can rewrite the above equation:

$$\frac{n(z, M_i)}{V_H} = \bar{F}_i(z, M_i) \frac{\rho_H}{\rho_H(t_{\text{in}})} \left(\frac{1}{\frac{4}{3}\pi(R_{\text{in}})^3} \right). \quad (10)$$

5. Results

In this section we compare two cases of collapse models: spherical (no kinematical backreaction) and generic; both cases start from Gaussian distributions in the initial invariants. Figure 1 (left panel) shows that the individual probabilities of collapse

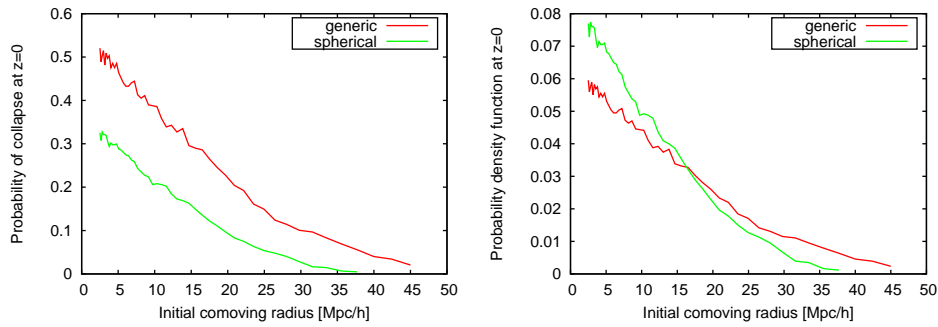


Fig. 1. Non-normalised probabilities and normalised (according to Eq. (8)) probability density function for the collapse of objects as a function of initial radius, at redshift $z = 0$.

are always higher for the generic case compared with the spherical model (this result has been also observed using different, non-spherical but less general approaches).

The shear and the domain-dependent expansion rate accelerate the collapse, allowing bigger structures to form. Figure 1 (right panel) shows that making the assumption that all mass resides in collapsed objects at redshift $z = 0$ changes the relation between these models—since bigger structures form in the generic case, less dark matter particles go into the low-mass end of the probability density function in comparison to the spherical case.

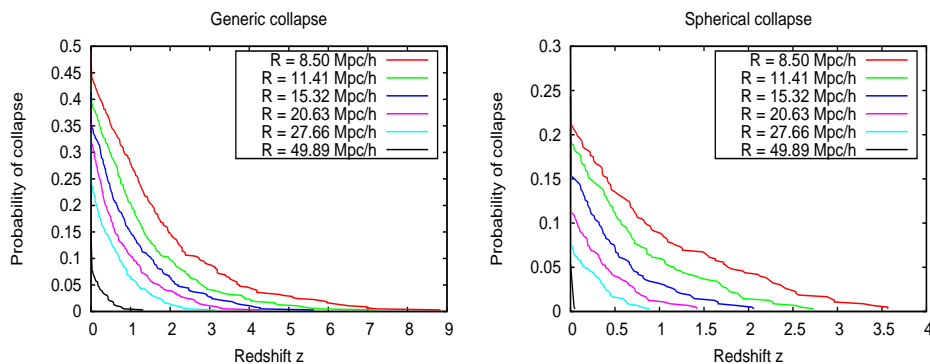


Fig. 2. Redshift-dependent probability of collapse for different mass scales.

6. Conclusions and Outlook

Spherical collapse, being oversimplified, does not provide a plausible analysis of structure formation compared with the generic situation presented here. However, although individual probabilities of collapse typically differ between the models by a factor of about two (1.72 at initial $R = 8$ Mpc/h comoving), when separately normalised to make all mass collapse by the present, this ratio drops to about unity. Taking into account the backreaction term results in higher abundances of collapsed objects at higher redshifts, and allows for a bigger bound structures to form (Fig. 2). However, predictions concerning the low-mass end of the probability density function have to be taken with caution, since, as stated above, a better matter model is required to properly access the low mass regime.

In a work in preparation [15] we also aim at understanding the role of curvature in the distribution of collapsed objects. We know from earlier considerations [16] that collapsed objects reside in positive-curvature environments. Since we include backreaction in the generic model, we can quantify the prediction that positive curvature energies add up to the effective matter source, providing a scale-dependent abundance of a component that would be interpreted as dark matter in the standard model. By assuming purely baryonic matter content in the initial power spectrum, instead of the normal assumption that the matter component is dominated by non-baryonic dark matter, the roles of matter content and curvature effects can be separated.

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